The transformed lamina stiffnesses, \( \overline{Q}_{ij} \), for the AS/3501 material are given in Example 7.3. Using these values, the extensional stiffnesses of the \([+45/-45]\) laminate are:

\[
\overline{A}_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k - z_{k-1}) = (\overline{Q}_{ij})_{+45} + t + (\overline{Q}_{ij})_{-45}
\]

where \( t = \) ply thickness = 0.25 mm

\[
\begin{align*}
\overline{A}_{11} &= (\overline{Q}_{11})_{+45} + t + (\overline{Q}_{11})_{-45} = 45.22 \times 0.25 + 45.22 \times 0.25 \\
\overline{A}_{22} &= 22.61 \text{ GPa} \cdot \text{mm} = \overline{A}_{22} \\
\overline{A}_{12} &= (\overline{Q}_{12})_{+45} + t + (\overline{Q}_{12})_{-45} = 31.42 \times 0.25 + 31.42 \times 0.25 \\
\overline{A}_{12} &= 15.71 \text{ GPa} \cdot \text{mm} = \overline{A}_{21} \\
\overline{A}_{16} &= \overline{A}_{26} = 0 \\
\overline{A}_{66} &= (\overline{Q}_{66})_{+45} + t + (\overline{Q}_{66})_{-45} = 35.6 \times 0.25 + 35.6 \times 0.75 \\
\overline{A}_{66} &= 17.8 \text{ GPa} \cdot \text{mm}
\end{align*}
\]

and \( \overline{[A]} = \begin{bmatrix} 22.61 & 15.71 & 0 \\ 15.71 & 22.61 & 0 \\ 0 & 0 & 17.8 \end{bmatrix} \text{ GPa} \cdot \text{mm} \)

The coupling stiffnesses are given by

\[
\overline{B}_{ij} = \frac{1}{t} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^2 - z_{k-1}^2)
\]

so that

\[
\begin{align*}
\overline{B}_{11} &= \frac{1}{t} \left[ (\overline{Q}_{11})_{+45} (z_1^2 - z_0^2) + (\overline{Q}_{11})_{-45} (z_2^2 - z_1^2) \right] \times 0.25 \\
\overline{B}_{11} &= 0 = \overline{B}_{22} \\
\overline{B}_{12} &= \frac{1}{t} \left[ (\overline{Q}_{12})_{+45} (0 - 0.25^2) + (\overline{Q}_{12})_{-45} (0.75^2 - 0) \right] = 0 = \overline{B}_{21}
\end{align*}
\]
\[ B_{16} = \frac{1}{2} \left[ 32.44 (0-0.25^2) - 32.44 (0, 0.25^2 - 0) \right] = -2.027 \text{ GPa} \cdot \text{mm}^2 \]
\[ B_{26} = \frac{1}{2} \left[ 32.44 (0-0.25^2) - 32.44 (0.25^2 - 0) \right] = -2.027 \text{ GPa} \cdot \text{mm}^2 \]
\[ B_{66} = 0. \]

and
\[
[B] = \begin{bmatrix}
0 & 0 & -2.027 \\
0 & 0 & -2.027 \\
-2.027 & -2.027 & 0
\end{bmatrix} \text{ GPa} \cdot \text{mm}^2
\]

The bending stiffnesses are given by
\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (Q_{ij})_k (z_{k}^3 - z_{k-1}^3)
\]

so that
\[
D_{11} = \frac{1}{3} \left[ (Q_{11}) (z_0^3 - z_1^3) + (Q_{11}) (z_1^3 - z_2^3) \right]
\]
\[
D_{11} = \frac{1}{3} \left[ 45.22 (0-0.25^3) + 45.22 (0.25^3-0) \right] = 0.471 \text{ GPa} \cdot \text{mm}^3
\]
\[ D_{22} = D_{11} = 0.471 \text{ GPa} \cdot \text{mm}^3 \]
\[ D_{12} = D_{21} = \frac{1}{3} \left[ 31.42 (0-0.25^3) + 31.42 (0.25^3-0) \right] \]
\[ D_{12} = D_{21} = 0.327 \text{ GPa} \cdot \text{mm}^3 \]
\[ D_{16} = D_{26} = 0 \]
\[ D_{66} = \frac{1}{2} \left[ 35.6 (0-0.25^3) + 35.6 (0.25^3-0) \right] \]
\[ D_{66} = 0.371 \text{ GPa} \cdot \text{mm}^3 \]

and
\[
[D] = \begin{bmatrix}
0.471 & 0.327 & 0 \\
0.327 & 0.471 & 0 \\
0 & 0 & 0.371
\end{bmatrix} \text{ GPa} \cdot \text{mm}^3
\]
The stiffnesses $A_{ij}$ associated with a particular orientation can be determined by adding the same angle to each ply orientation. For example, the $[-60/0/60]$ laminate can be analyzed as a $[E50/10/70]$ laminate by adding 10° to each ply orientation to simulate the stiffness associated with loading along the 10° orientation. Repeating this procedure for 10° increments, the following results can be obtained, where all $A_{ij}$ are in units of $10^8$ Pa.m.

<table>
<thead>
<tr>
<th>Lamine</th>
<th>$A_{11}$</th>
<th>$A_{22}$</th>
<th>$A_{12}$</th>
<th>$A_{66}$</th>
<th>$A_{16}$, $A_{26}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-60/0/60]$</td>
<td>2.29</td>
<td>2.29</td>
<td>0.678</td>
<td>0.806</td>
<td>0</td>
</tr>
<tr>
<td>$[-50/10/70]$</td>
<td>2.29</td>
<td>2.29</td>
<td>0.678</td>
<td>0.806</td>
<td>0</td>
</tr>
<tr>
<td>$[-40/20/80]$</td>
<td>2.29</td>
<td>2.29</td>
<td>0.678</td>
<td>0.806</td>
<td>0</td>
</tr>
<tr>
<td>$[-30/30/90]$</td>
<td>2.29</td>
<td>2.29</td>
<td>0.678</td>
<td>0.806</td>
<td>0</td>
</tr>
<tr>
<td>$[30/90/150]$</td>
<td>2.29</td>
<td>2.29</td>
<td>0.678</td>
<td>0.806</td>
<td>0</td>
</tr>
</tbody>
</table>

The results are plotted as functions of the incremented angle, or loading angle, below.

![Plot of stiffnesses vs. loading angle](image-url)
Similar calculations for a $[0/45/90]$ laminate yield the results shown in the graph below.

Conclusions: The $A_{ij}$ are independent of orientation for the $[-60/0/60]$ laminate, but not for the $[0/45/90]$ laminate. Thus, the $[-60/0/60]$ laminate is quasi-isotropic, but the $[0/45/90]$ is not.
The simplest approach is to use a computer program such as the ones listed in Table 7.1 to calculate the resulting strain $\varepsilon_x$ for a fixed $\Delta T$ input and various ply orientations $[\pm \theta/\pm \theta]$. By plotting $\varepsilon_x$ vs. the angle $\theta$, the angle $\theta$ where $\varepsilon_x$ is a minimum can be found, and the coefficient of thermal expansion

$$\alpha_x = \frac{|\varepsilon_x|}{\Delta T}$$

would also be a minimum at the same angle. Assuming $\Delta T = 100^\circ K$ and repeating this procedure a sufficient number of times, the plot below shows that $|\varepsilon_x|$ and therefore $\alpha_x$, goes to zero somewhere in the range $8.55^\circ < \theta < 8.6^\circ$.

Recall that, in Problem 5.10, it was found that $\alpha_x$ goes to zero at an angle of $\theta=5.9^\circ$ for a single lamina of a quite similar material. The reason for the difference between the two solutions is that the plies interact in the $[\pm \theta/\pm \theta]$ laminate, but there is obviously no ply interaction in the case of the single ply.
From Example 7.10, the lamina stiffness matrix:

\[ [Q] = [\bar{Q}]_{oo} = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 9.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{ GPa} \]

\[ [\bar{Q}] = \begin{bmatrix} 9.05 & 2.72 & 0 \\ 2.72 & 138.8 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{ GPa} \]

The extensional stiffness for the \([0/90/0]_5\) laminate is:

\[ A_{ij} = 4(0.25) [\bar{Q};ij]_{oo} + 2(0.25) [\bar{Q};ij]_{90^o} \]

\[ A_{ij} = [\bar{Q};ij]_{oo} + 1.5 [\bar{Q};ij]_{90^o} \]

or, for the first increment before first ply failure,

\[ [A]^{(1)} = \begin{bmatrix} 143.3 & 4.08 & 0 \\ 4.08 & 78.45 & 0 \\ 0 & 0 & 10.35 \end{bmatrix} \text{ GPa - mm} \]

The lamina failure strains are:

\[ \varepsilon_L^{(4)} = \frac{S_L^{(4)}}{E_1} = \frac{1448}{138(10^3)} = 0.0105 \]

\[ \varepsilon_T^{(4)} = \frac{S_T^{(4)}}{E_2} = \frac{48.3}{9(10^3)} = 0.0054 \]

First ply failure occurs at \( \varepsilon_x^{(1)} = \varepsilon_T^{(4)} = 0.0054 \) and the laminate force-deformation equations can be written as:

\[ \begin{bmatrix} N_x^{(1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 143.3 & 4.08 & 0 \\ 4.08 & 78.45 & 0 \\ 0 & 0 & 10.35 \end{bmatrix} \begin{bmatrix} \varepsilon_x^{(1)} \\ \gamma_{xx}^{(1)} \end{bmatrix} \text{ (continued)} \]
Solving these equations simultaneously, we get
\[ N_x^{(2)} = 0.773 \text{ GPa} \cdot \text{mm}, \quad e_y^{(2)} = -0.000281, \quad \gamma_{xy}^{(2)} = 0 \]

Now we adjust \([A]\) to account for first ply failure.

As in Example 7.10, we will try two different approaches.

(a) In the first approach, we set \(\bar{Q}_{ij}^{(2)} = 0\) so that the adjusted laminate stiffness matrix \(A\)'s

\[
\begin{bmatrix}
A^{(2)}
\end{bmatrix} = \begin{bmatrix} \bar{Q}_{ij} \end{bmatrix}_{0} = \begin{bmatrix}
138.8 & 2.72 & 0 \\
2.72 & 9.05 & 0 \\
0 & 0 & 6.9
\end{bmatrix} \text{ GPa} \cdot \text{mm}
\]

As in Example 7.10, the second strain increment is
\[ e_x^{(2)} = e_{x}^{(1)} - e_x^{(1)} = 0.0105 - 0.0054 = 0.0051 \]

and the incremental force-deformation equations become

\[
\begin{bmatrix}
N_x^{(2)} \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
138.8 & 2.72 & 0 \\
2.72 & 9.05 & 0 \\
0 & 0 & 6.9
\end{bmatrix} \begin{bmatrix}
e_y^{(2)} \\
e_y^{(2)} \\
\gamma_{xy}^{(2)}
\end{bmatrix}
\]

Solving simultaneously,

\[ N_x^{(2)} = 0.704 \text{ GPa} \cdot \text{mm}, \quad e_y^{(2)} = -0.00153, \quad \gamma_{xy}^{(2)} = 0 \]

and the ultimate failure load of the laminate is

\[ N_x^{\text{Total}} = N_x^{(1)} + N_x^{(2)} = 0.773 + 0.704 = 1.477 \text{ GPa} \cdot \text{mm} \]

(b) In the second approach, we set \(E_2 = G_{12} = \gamma_{21} = 0\) for the failed 90° plies, but we assume that \(E_1\) is unchanged. Thus,
(continued)

\[ [Q_{22}]_{q0} = E_1 = 138 \text{ GPa} \]
\[ [Q_{11}]_{q0} = [Q_{12}]_{q0} = [Q_{66}]_{q0} = 0 \]

and

\[
[A^{(2)}] = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 78.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{ GPa} \cdot \text{mm}
\]

so that

\[
\begin{bmatrix} N_x^{(2)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 78.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \begin{bmatrix} 0.0051 \\ E_y^{(2)} \\ \gamma_{xy}^{(2)} \end{bmatrix}
\]

Solving simultaneously,

\[ N_x^{(2)} = 0.707 \text{ GPa} \cdot \text{mm}, \quad E_y^{(2)} = -0.000178, \quad \gamma_{xy}^{(2)} = 0 \]

and the laminate ultimate failure load is

\[ N_{x_{\text{total}}^{(2)}} = N_x^{(1)} + N_x^{(2)} = 0.773 + 0.707 = 1.480 \text{ GPa} \cdot \text{mm} \]

The results from the two approaches are shown in the plot below. As in Example 7.10, approach (a) is more conservative than approach (b).